

DIFFERENTIAL GRADED CONTACT GEOMETRY AND JACOBI STRUCTURES

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ABSTRACT. We study contact structures on nonnegatively-graded manifolds equipped with homological contact vector fields. In the degree 1 case, we show that there is a one-to-one correspondence between such structures and Jacobi manifolds. This correspondence allows us to reinterpret the Poissonization procedure, taking Jacobi manifolds to Poisson manifolds, as a supergeometric version of symplectization.

1. INTRODUCTION

A manifold whose algebra of functions is equipped with a *local Lie algebra* structure, in the sense of Kirillov [Kir76], is called a *Jacobi manifold*. Equivalently, a Jacobi manifold is a manifold equipped with a bivector field Λ and a vector field R , satisfying the equations (14). The definition of Jacobi manifolds in these terms is due to Lichnerowicz [Lic77, Lic78], who viewed it as a “contravariant generalization of the notion of contact manifold.” Since Poisson manifolds form the covariant generalization of the notion of symplectic manifolds, Lichnerowicz’ claim may be concisely described by the following analogy:

(1) Jacobi : contact :: Poisson : symplectic

The following known results provide ways of formalizing this analogy:

- There is a *Poissonization* process, taking Jacobi manifolds to Poisson manifolds [Lic78]. This parallels the symplectization process that takes contact manifolds (with contact 1-form) to symplectic manifolds.
- Jacobi manifolds “integrate” to *contact groupoids* [KSB93, CZ07]. This parallels the fact that Poisson manifolds integrate to symplectic groupoids [CDW87].

Note that these two approaches are “orthogonal” to each other, in terms of the analogy (1).

Although the latter approach is very interesting and potentially useful in many ways, it also has complications. Specifically (and in correspondence with the right side of analogy (1)), not every Jacobi manifold is integrable, and for those that are, it may be all but impossible to explicitly perform the integration.

The purpose of this paper is to describe another way of connecting Jacobi and contact structures, allowing us to formalize the analogy (1) without the difficulties of integration. Namely, we show that there is a one-to-one correspondence between Jacobi manifolds and degree 1 contact NQ -manifolds. This result parallels

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the Ševera-Roytenberg correspondence [Šev05, Roy02a] between Poisson manifolds and degree 1 symplectic $\mathbb{N}Q$ -manifolds. We furthermore show that, in this “supergeometric” point of view, Poissonization is *the same thing* as symplectization in the $\mathbb{N}Q$ category. In other words, the following diagram commutes:

$$(2) \quad \begin{array}{ccc} \boxed{\text{Jacobi manifolds}} & \longleftrightarrow & \boxed{\text{Deg. 1 contact } \mathbb{N}Q\text{-manifolds}} \\ \text{Poissonization} \downarrow & & \downarrow \text{Symplectization} \\ \boxed{\text{Poisson manifolds}} & \longleftrightarrow & \boxed{\text{Deg. 1 symplectic } \mathbb{N}Q\text{-manifolds}} \end{array}$$

The correspondence between Poisson manifolds and degree 1 symplectic $\mathbb{N}Q$ -manifolds has led to interesting results relating to Poisson reduction [CZ09, Meh11]. It also clarifies the relation between integration and quantization [CF01], via the AKSZ formalism [ASZK97]. We believe that the correspondence between Jacobi manifolds and degree 1 contact $\mathbb{N}Q$ -manifolds should lead to analogous results. We plan to explore these ideas elsewhere.

Although the emphasis of this paper is on the degree 1 case, we develop much of the general theory of contact $\mathbb{N}Q$ -manifolds in arbitrary degree. We remark that the degree 2 case should provide a natural generalization of Courant algebroids, together with a “Courantization” process. This approach may be useful in studying Jacobi-Dirac and generalized contact structures [Wad00, IPW05, PW10].

The existence of a correspondence between Jacobi manifolds and degree 1 contact $\mathbb{N}Q$ -manifolds was known by Ševera, who mentioned it in a footnote of [Šev05], but did not provide any details. More recently, Antunes and Laurent-Gengoux [ALG11] studied Jacobi bialgebroid structures from the supergeometric point of view. There are certainly relations between their results and ours, but neither is a special case of the other. Additionally, contact structures on supermanifolds were considered by Bruce [Bru11a, Bru11b]. His papers played a role in inspiring the author to consider contact $\mathbb{N}Q$ -manifolds.

Shortly after a preprint version of this paper appeared on the arXiv, a preprint by Grabowski [Gra11] appeared, covering similar material, but with emphasis on the degree 2 case. His work partially fulfills the above suggestion of a natural generalization of Courant algebroids. However, we should emphasize that there is an important distinction between his approach and ours. He identifies a graded contact manifold with its symplectization (which is larger but carries an \mathbb{R}^\times -action), and he develops the theory completely in terms of this identification. On the other hand, we show (Corollary 2.7) that a contact \mathbb{N} -manifold of degree $n > 0$ can be identified with a symplectic \mathbb{N} -manifold that is one dimension *smaller* in degree n , and our results are stated in terms of this correspondence. The processes of reduction (by the \mathbb{R}^\times -action) and symplectization will allow one to translate between Grabowski’s framework and ours.

1.1. Conventions. There are now many good introductions to the theory of \mathbb{Z} - and \mathbb{N} -graded manifolds, including [Meh06, Vor02, Roy02b, CS11] (although, in contrast to [Vor02], we adopt a definition for which a function’s parity agrees with its weight or degree). Roytenberg’s paper [Roy02b] is particularly relevant, since it contains the details of the Ševera-Roytenberg correspondence, which plays an important role in motivating this paper. We will freely use his results on symplectic \mathbb{N} -manifolds without making explicit reference.

There are many possible sign conventions for the calculus of differential forms on a graded manifold. We will use the conventions of [Meh06, Meh09], where the algebra $\Omega(\mathcal{M})$ of differential forms on a graded manifold \mathcal{M} consists, by definition, of polynomial functions on $T[1]\mathcal{M}$. Thus, the algebra $\Omega(\mathcal{M})$ is graded-commutative with respect to the total grading (i.e. the sum of the “form” grading and the internal “manifold” grading). When we say that a p -form is of degree k , we mean that the manifold grading is k .

With this choice of sign convention, the Cartan commutation relations include the following identities for any homogeneous vector fields X, Y on \mathcal{M} :

$$\begin{aligned}\mathcal{L}_X &= [\iota_X, d] = \iota_X d + (-1)^{|X|} d \iota_X, \\ \iota_{[X, Y]} &= [\mathcal{L}_X, \iota_Y] = \mathcal{L}_X \iota_Y - (-1)^{|X|(|Y|-1)} \iota_Y \mathcal{L}_X, \\ \iota_X \iota_Y &= (-1)^{(|X|-1)(|Y|-1)} \iota_Y \iota_X.\end{aligned}$$

On graded symplectic manifolds, we take Hamiltonian vector fields to be defined by the equation $df = (-1)^{|X|-1} \iota_X \omega$. Note that, if the degree of the symplectic form ω is n , then $|X| = |f| - n$. Poisson brackets are given by $\{f, g\} = X(g) = (-1)^{|Y|-1} \iota_X \iota_Y \omega$, where X and Y are the Hamiltonian vector fields associated to f and g , respectively. The reader may verify that this convention gives the correct skew-commutativity rule for a degree $-n$ Lie bracket.

2. CONTACT N -MANIFOLDS

In this section, we give the definition and some basic properties of degree n contact N -manifolds. Most of the results are straightforward extensions of well-known results from ordinary contact geometry. There are two features that are unique to the graded case. The first is the appearance of the Euler vector field, which automatically preserves the contact structure. The second is the fact (see Theorem 2.6) that, when $n > 0$, any degree n contact N -manifold naturally splits as the product of $\mathbb{R}[n]$ and a degree n symplectic N -manifold. This splitting gives a one-to-one correspondence between contact N -manifolds and symplectic N -manifolds of degree $n > 0$.

2.1. Definition. Let \mathcal{M} be an N -graded manifold (or N -manifold, for short), and let α be a nowhere-vanishing 1-form of degree n on \mathcal{M} . The assignment $X \mapsto \iota_X \alpha$ is (left) $C^\infty(\mathcal{M})$ -linear and so defines a degree $-n$ bundle map

$$\iota.\alpha: T\mathcal{M} \rightarrow \mathcal{M} \times \mathbb{R}.$$

The kernel $\mathcal{D} := \ker \iota.\alpha$ is a distribution of corank 1 concentrated in degree n .

We say that α is a *contact form* if $d\alpha$ induces a nondegenerate pairing on \mathcal{D} . Since $d\alpha$ is a degree n 2-form, the nondegeneracy requirement imposes the same restrictions on the rank of \mathcal{D} that one sees on the dimensions of degree n symplectic N -manifolds, namely that $\text{rank}_i \mathcal{D} = \text{rank}_{n-i} \mathcal{D}$. This implies that $\dim_0 \mathcal{M} = \dim_n \mathcal{M} - 1$, and that $\dim_i \mathcal{M} = \dim_{n-i} \mathcal{M}$ for $i > 0$.

The following statements are straightforward consequences of the definitions.

Lemma 2.1. *Let α be a contact form. Then*

- (1) $T^*\mathcal{M} = \text{im}(d\alpha)^\flat \oplus \langle \alpha \rangle$, and
- (2) *the degree n map $\mathfrak{X}(\mathcal{M}) \rightarrow \Gamma(\text{im}(d\alpha)^\flat) \oplus C^\infty(\mathcal{M})$, $X \mapsto (\iota_X d\alpha, \iota_X \alpha)$ is an isomorphism of left $C^\infty(\mathcal{M})$ -modules.*

A *contact \mathbb{N} -manifold of degree n* is an \mathbb{N} -manifold equipped with a contact form α . We note that this definition is somewhat against spirit of contact geometry, where the fundamental structure is taken to be the distribution \mathcal{D} , and therefore the contact form is only well-defined up to a nonvanishing function. We plan to elaborate on this ambiguity elsewhere, but for the present purposes we assume the contact form to be fixed.

2.2. Contact vector fields. Let (\mathcal{M}, α) be a degree n contact \mathbb{N} -manifold. A vector field $X \in \mathfrak{X}(\mathcal{M})$ is *contact* if

$$(3) \quad \mathcal{L}_X \alpha = (-1)^{|X|} f \alpha$$

for some $f \in C^\infty(\mathcal{M})$. The sign in (3) is only there to simplify the signs in later formulae.

We will now describe the contact analogue of Hamiltonian vector fields. Let h be a (homogeneous) function on \mathcal{M} . Then, by Lemma 2.1, we may uniquely write

$$(4) \quad dh = \beta + f \alpha,$$

where $\beta \in \text{im}(d\alpha)^\flat$ and $f \in C^\infty(\mathcal{M})$. Again by Lemma 2.1, there exists a unique vector field X such that

$$(5) \quad \iota_X d\alpha = (-1)^{|h|-n+1} \beta, \quad \iota_X \alpha = h.$$

In this case, we have that $|X| = |h| - n$. Then

$$\begin{aligned} \mathcal{L}_X \alpha &= \iota_X d\alpha + (-1)^{|X|} d\iota_X \alpha \\ &= (-1)^{|X|+1} \beta + (-1)^{|X|} dh \\ &= (-1)^{|X|} f \alpha, \end{aligned}$$

so X is contact.

The process taking functions to contact vector fields is invertible; given a contact vector field X , one can recover a function h via (5), and the contact vector field associated to h is again X . In summary, we have the following:

Proposition 2.2. *There is a one-to-one correspondence between functions and contact vector fields on \mathcal{M} . Functions of degree k correspond to contact vector fields of degree $k - n$.*

Example 2.3. The *Reeb vector field* ρ is the degree $-n$ vector field defined by the equations $\iota_\rho d\alpha = 0$ and $\iota_\rho \alpha = 1$. Under the correspondence of Proposition 2.2, the Reeb vector field corresponds to the constant function 1.

When n is odd, we have that $[\rho, \rho] = 2\rho^2$ automatically vanishes, since there are no nontrivial degree $-2n$ vector fields on \mathcal{M} . When $n > 0$ is even, we have, again by degree considerations, that the formal power series $\exp(\rho)(f)$ is a finite sum for any function f . In other words, ρ is both integrable and complete.

The Reeb vector field allows us to explicitly perform the decomposition (4). Note that, if $\beta \in \text{im}(d\alpha)^\flat$, then $\iota_\rho \beta = 0$. Thus, for any $h \in C^\infty(\mathcal{M})$, with β and f given by (4), we have that

$$(6) \quad \rho(h) = \iota_\rho dh = \iota_\rho f \alpha = (-1)^{(n-1)|f|} f.$$

This allows us to solve for β :

$$(7) \quad \beta = dh - (-1)^{(n-1)|h|} \rho(h) \alpha.$$

Example 2.4. The Euler vector field ε is the degree 0 vector field given by $\varepsilon(f) = |f|f$ for any homogeneous function f . Since α is of degree n , we have that $\mathcal{L}_\varepsilon \alpha = n\alpha$, so the Euler vector field is contact. Let $\theta := \iota_\varepsilon \alpha$ be called the Euler function of (\mathcal{M}, α) . The degree of θ is n .

Lemma 2.5. *The Reeb vector field and the Euler function satisfy the equation $\rho(\theta) = n$.*

Proof. Using the definition of θ , we have that

$$\begin{aligned} \rho(\theta) &= \mathcal{L}_\rho \iota_\varepsilon \alpha \\ &= \iota_\rho \mathcal{L}_\varepsilon \alpha + \iota_\rho \iota_\varepsilon d\alpha. \end{aligned}$$

The latter term vanishes because $\iota_\rho d\alpha = 0$, and the first term is $n\iota_\rho \alpha = n$. \square

2.3. The structure of contact \mathbb{N} -manifolds. Let (\mathcal{M}, α) be a degree n contact \mathbb{N} -manifold where $n > 0$. As we will see, the assumption $n > 0$ makes the situation drastically different from the case of ordinary contact manifolds.

Let $\lambda := \alpha - \frac{1}{n}d\theta$ and $\omega := d\lambda = d\alpha$.

Theorem 2.6. *Let \mathcal{M} be a degree n contact \mathbb{N} -manifold, where $n > 0$. Then \mathcal{M} is the total space of a principal $\mathbb{R}[n]$ -bundle with a canonical trivialization. The 1-form λ is basic, and ω passes to a degree n symplectic form on the quotient.*

Proof. Lemma 2.5 implies that ρ is nowhere vanishing. This, together with the fact that ρ is integrable and complete, gives a free $\mathbb{R}[n]$ action on \mathcal{M} , and we take \mathcal{N} to be the quotient. The graded algebra of functions on \mathcal{N} consists of those functions f on \mathcal{M} for which $\rho(f) = 0$. On the other hand, the function θ determines a projection map $\mathcal{M} \rightarrow \mathbb{R}[n]$ that trivializes the principal $\mathbb{R}[n]$ -bundle $\mathcal{M} \rightarrow \mathcal{N}$.

Using Lemma 2.5, we see that $\iota_\rho \lambda = 0$ and $\mathcal{L}_\rho \lambda = 0$, so λ is basic. Nondegeneracy of the push-forward of ω to \mathcal{N} follows from the fact that ρ spans the characteristic distribution for the presymplectic form $d\alpha$. \square

In Theorem 2.6, the contact form on \mathcal{M} can be recovered from the symplectic form ω on \mathcal{N} , since

$$(8) \quad \lambda = \frac{1}{n} \iota_\varepsilon \omega \quad \text{and} \quad \alpha = \lambda + \frac{1}{n} d\theta.$$

Furthermore, given any degree n symplectic \mathbb{N} -manifold (\mathcal{N}, ω) for $n > 0$, equations (8) define a degree n contact form on $\mathcal{N} \times \mathbb{R}[n]$. We therefore have the following result:

Corollary 2.7. *When $n > 0$, there is a one-to-one correspondence between degree n symplectic \mathbb{N} -manifolds and degree n contact \mathbb{N} -manifolds.*

In particular, every degree 1 contact \mathbb{N} -manifold is canonically of the form $T^*[1]M \times \mathbb{R}[1]$ where M is a manifold. In this case, the contact form is $\alpha = \lambda + d\theta$, where λ is the Liouville 1-form on $T^*[1]M$, and θ is the coordinate function on $\mathbb{R}[1]$. The Reeb vector field is $\rho = \frac{\partial}{\partial \theta}$.

3. CONTACT $\mathbb{N}Q$ -MANIFOLDS

Recall that a *homological vector field* on a graded manifold \mathcal{M} is a degree 1 vector field Q such that $Q^2 = 0$. If \mathcal{M} has a contact form, then we may consider vector fields that are both contact and homological.

Definition 3.1. A degree n contact $\mathbb{N}Q$ -manifold is a degree n contact \mathbb{N} -manifold \mathcal{M} , equipped with a vector field Q that is contact and homological.

3.1. The case $n = 1$. In this section, we show that degree 1 contact $\mathbb{N}Q$ -manifolds are in one-to-one correspondence with Jacobi manifolds.

Recall from §2.3 that every degree 1 contact \mathbb{N} -manifold is canonically of the form $\mathcal{M} = T^*[1]M \times \mathbb{R}[1]$, for some ordinary manifold M . We remind the reader that functions on $T^*[1]M$ can be identified with multivector fields on M .

Let us first describe degree 1 contact vector fields on \mathcal{M} . By Proposition 2.2, every degree 1 contact vector field arises from a degree 2 function h on \mathcal{M} . Any such function is of the form

$$h = \Lambda + \theta R,$$

where Λ is a bivector field and R is a vector field on M . Following (4), (6), and (7), we write

$$(9) \quad dh = d\Lambda - \theta dR + R d\theta = d\Lambda - \theta dR - R\lambda + R\alpha,$$

where $\beta := d\Lambda - \theta dR - R\lambda$ is in $\text{im}(d\alpha)^\flat$. The corresponding contact vector field Q is defined by the equations (5), which in this case become

$$(10) \quad \iota_Q \alpha = \Lambda + \theta R,$$

$$(11) \quad \iota_Q d\alpha = d\Lambda - \theta dR - R\lambda.$$

The unique solution is

$$(12) \quad Q = X_\Lambda + \theta X_R - R\varepsilon - (\Lambda + \theta R) \frac{\partial}{\partial \theta},$$

where X_Λ and X_R are the Hamiltonian vector fields on $T^*[1]M$ associated to Λ and R , respectively. The Hamiltonian vector fields annihilate θ and act on multivector fields via the Schouten bracket.

The verification of (12) is a straightforward exercise, using the definition of Hamiltonian vector fields and (8). Applying Proposition 2.2, we have the following result:

Proposition 3.2. *Every degree 1 contact vector field Q on \mathcal{M} is of the form (12) for some $\Lambda \in \mathfrak{X}^2(M)$ and $R \in \mathfrak{X}^1(M)$.*

Comparing (9) with the construction of §2.2, we have that

$$(13) \quad \mathcal{L}_Q \alpha = -R\alpha.$$

Next, we consider the conditions on Λ and R that arise from the requirement $Q^2 = 0$.

Proposition 3.3. *Let Q be a contact vector field of the form (12). Then $Q^2 = 0$ if and only if*

$$(14) \quad [\Lambda, \Lambda] = 2R\Lambda \quad \text{and} \quad [R, \Lambda] = 0.$$

Proof. Contact vector fields are closed under the Lie bracket, so $Q^2 = \frac{1}{2}[Q, Q]$ is contact. By the correspondence of Proposition 2.2, we have that $Q^2 = 0$ if and only if $\iota_{[Q, Q]} \alpha = 2\iota_{Q^2} \alpha = 0$. Using (10), (12), and (13), we then compute

$$\begin{aligned} \iota_{[Q, Q]} \alpha &= \mathcal{L}_Q \iota_Q \alpha - \iota_Q \mathcal{L}_Q \alpha \\ &= \mathcal{L}_Q (\Lambda + \theta R) + \iota_Q (R\alpha) \\ &= [\Lambda, \Lambda] - 2R\Lambda + 2\theta[R, \Lambda], \end{aligned}$$

which vanishes if and only if the equations (14) hold. \square

The equations (14) are exactly those that define a Jacobi structure on M . Thus, we have shown the following:

Theorem 3.4. *There is a one-to-one correspondence between Jacobi manifolds and degree 1 contact $\mathbb{N}Q$ -manifolds.*

Remark 3.5. It is well-known [KSB93, Vai00] that one can associate to a Jacobi manifold M a Lie algebroid structure on $T^*M \times \mathbb{R}$. The search for a converse result, characterizing those Lie algebroid structures on $T^*M \times \mathbb{R}$ that arise from Jacobi structures, led Iglesias and Marrero [IM01] and Grabowski and Marmo [GM01] to define the notion of *Jacobi bialgebroids*.

We can interpret Theorem 3.4 as giving a more direct answer to the same question. Namely, the Lie algebroid structures on $T^*M \times \mathbb{R}$ that arise from Jacobi structures are exactly those for which the Lie algebroid differential is a contact vector field.

4. SYMPLECTIZATION

Let (\mathcal{M}, α) be a degree n contact \mathbb{N} -manifold. On $\mathcal{M} \times \mathbb{R}$, one defines a 2-form $\tilde{\omega} = d(e^t \alpha) = e^t(dt \cdot \alpha + d\alpha)$. Since the coordinate t on \mathbb{R} is of degree 0, we have that $\tilde{\omega}$ is of degree n . The assumptions on α imply that $\tilde{\omega}$ is nondegenerate, so $\mathcal{M} \times \mathbb{R}$ is a degree n symplectic manifold. The process taking (\mathcal{M}, α) to $(\mathcal{M} \times \mathbb{R}, \tilde{\omega})$ is called *symplectization*.

The following lemmas describe the relationship between contact vector fields and the symplectization process.

Lemma 4.1. *Let $X \in \mathfrak{X}(\mathcal{M})$ be a contact vector field, and let f be the corresponding function in (3). Then $X - f \frac{\partial}{\partial t}$ is a Hamiltonian vector field on $\mathcal{M} \times \mathbb{R}$, with Hamiltonian function $H_X := e^t(\iota_X \alpha)$.*

Proof. On the one hand, we have that

$$dH_X = d(e^t \iota_X \alpha) = e^t(dt \cdot \iota_X \alpha + d\iota_X \alpha).$$

On the other hand, we have that

$$\iota_{X - f \frac{\partial}{\partial t}} \tilde{\omega} = e^t \left((-1)^{|X|-1} dt \cdot \iota_X \alpha + \iota_X d\alpha - f\alpha \right).$$

The conclusion follows from (3) and the identity $\mathcal{L}_X = \iota_X d + (-1)^{|X|} d\iota_X$. \square

Lemma 4.2. *Let $Q \in \mathfrak{X}(\mathcal{M})$ be a homological contact vector field, and let φ be the degree 1 function such that $\mathcal{L}_Q \alpha = -\varphi \alpha$. Then the Hamiltonian vector field $Q - \varphi \frac{\partial}{\partial t}$ on $\mathcal{M} \times \mathbb{R}$ is also homological.*

Proof. On the one hand, $(\mathcal{L}_Q)^2 \alpha = \mathcal{L}_Q(-\varphi \alpha) = -(Q(\varphi))\alpha$. In the last step, we have used the fact that $\varphi^2 = 0$. On the other hand, $(\mathcal{L}_Q)^2 \alpha = \mathcal{L}_{Q^2} \alpha = 0$, since Q is homological. It follows that $Q(\varphi) = 0$.

Now, we can directly see that $(Q - \varphi \frac{\partial}{\partial t})^2 = Q^2 - Q(\varphi) \frac{\partial}{\partial t} = 0$. \square

Together, Lemmas 4.1 and 4.2 give the following result.

Theorem 4.3. *The symplectization process takes contact $\mathbb{N}Q$ -manifolds to symplectic $\mathbb{N}Q$ -manifolds.*

4.1. Poissonization. We now return to the case $n = 1$, where $\mathcal{M} = T^*[1]M \times \mathbb{R}[1]$. The symplectization process gives $T^*[1]M \times \mathbb{R}[1] \times \mathbb{R}$, with the symplectic form $\tilde{\omega} = e^t(dt(d\theta + \lambda) + \omega)$, where λ and ω are, respectively, the Liouville 1-form and the canonical symplectic form on $T^*[1]M$.

Given a Jacobi structure (Λ, R) on M , we have a homological contact vector field Q , given by (12). Lemmas 4.1 and 4.2 tell us that Q induces a homological Hamiltonian vector field on the symplectization $T^*[1]M \times \mathbb{R}[1] \times \mathbb{R}$, with Hamiltonian function $H_Q = e^t(\iota_Q \alpha) = e^t(\Lambda + \theta R)$; here, we have used (10).

In order to realize H_Q as a bivector field on $M \times \mathbb{R}$, we need to transform $\tilde{\omega}$ into the canonical symplectic form $\omega + dtd\theta$, arising from the obvious identification of $T^*[1]M \times \mathbb{R}[1] \times \mathbb{R}$ with $T^*[1](M \times \mathbb{R})$.

Consider the diffeomorphism ξ of $T^*[1]M \times \mathbb{R}[1] \times \mathbb{R}$, given by

$$\xi^* f = \exp(t\varepsilon)(f) = e^{|f|t} f$$

for any homogeneous function f . Using the power series expansion of the exponential, we can see that, for any homogeneous differential form β ,

$$\xi^* \beta = \exp(\mathcal{L}_{t\varepsilon})(\beta) = e^{|\beta|t} (\beta + dt \iota_\varepsilon \beta).$$

In particular, $\xi^* \omega = e^t(\omega + dt \cdot \lambda)$, and $\xi^*(dtd\theta) = e^t dtd\theta$. Thus, $\xi^*(\omega + dtd\theta) = \tilde{\omega}$. In other words:

Proposition 4.4. *The diffeomorphism ξ relates the symplectic form $\tilde{\omega}$ with the canonical symplectic form on $T^*[1](M \times \mathbb{R})$.*

Since H_Q is of degree 2, we have that

$$\xi_* H_Q = e^{-t}(\Lambda + \theta R),$$

which exactly corresponds to the bivector field for the Poissonization of the Jacobi structure (Λ, R) (since θ is the conjugate variable that plays the role of $\frac{\partial}{\partial t}$). Thus we have shown the following.

Theorem 4.5. *The diagram (2) commutes.*

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